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On Holomorphic Foliations in Complex Surfaces Transverse to a Sphere

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Abstract. We prove that a holomorphic foliation $\mathcal{F}$ on a Stein surface transverse to the boundary of a 4-ball is conjugated inside the ball to the foliation generated by the holomorphic vector field $z\frac{\partial}{\partial z}+(z+w)\frac{\partial}{\partial w}$, provided that the transversely holomorphic flow induced by $\mathcal{F}$ on the boundary of the ball has a parabolic closed orbit. The proof contains a classification of transversely holomorphic flows on 3-manifolds with a parabolic closed orbit.

Introduction

Let X be a Stein surface and let $\Omega \subset X$ be a relatively compact domain diffeomorphic to a 4-ball and with boundary $M = \partial\Omega$ smooth and diffeomorphic to a 3-sphere. Let $\mathcal{F}$ be a holomorphic foliation (with isolated singularities) on X, of complex dimension one, and assume that $\mathcal{F}$ is transverse to M. Examples of this situation are obtained taking as Ω a (suitable) small neighborhood of a singularity of a foliation in the Poincaré domain [**A**]; one may conjecture that, up to diffeomorphism, these are the only examples. Here we prove a theorem which supports that conjecture (see [**BS**] for an analogous result).

The foliation $\mathcal{F}$ induces on M a transversely holomorphic one dimensional foliation $\mathcal{L}$. We shall say that a closed leaf $\gamma \simeq \mathbf{S}^1$ of $\mathcal{L}$ is *parabolic* if a generator $h\colon (\mathbb{C},0) \to (\mathbb{C},0)$ of its holonomy satisfies $h'(0) = 1$, $h''(0) \neq 0$. For example, the foliation in $\mathbb{C}^2$ generated by the vector field

$$z\frac{\partial}{\partial z} + (z+w)\frac{\partial}{\partial w}$$

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is transverse to $\{(z,w) \in \mathbb{C}^2 | |z|^2 + |w|^2 = 1\}$ and $\{(z,w) \in \mathbb{C}^2 | z = 0, |w| = 1\}$ is a parabolic closed leaf of the induced foliation.

Theorem. *Let X, Ω, M, $\mathcal{F}$, $\mathcal{L}$ be as above and suppose that $\mathcal{L}$ has a parabolic closed leaf. Assume also that $H^2(X, \mathbb{Z}) = 0$. Then there exists a diffeomorphism $\phi: \overline{\Omega} \to \{(z,w) \in \mathbb{C}^2 | |z|^2 + |w|^2 \leq 1\}$ which sends the foliation $\mathcal{F}$ to the foliation $\mathcal{G}$ generated by the vector field*

$$z\frac{\partial}{\partial z} + (z+w)\frac{\partial}{\partial w}.$$

This diffeomorphism can be chosen transversely holomorphic.

The first part of the proof of this theorem consists in a classification of transversely holomorphic foliations on closed 3-manifolds possessing a parabolic closed leaf. There are two main ingredients: a theorem of Ghys and Gomez-Mont [**GG**], which gives the global behaviour of the foliation, and Écalle-Voronin results [**V**] [**MR**] about classification of germs of holomorphic diffeomorphisms tangent to the identity. The final result is that $\mathcal{L}$ is transversely holomorphically conjugate to the foliation induced by $\mathcal{G}$ on the unit sphere. This is used in the second part of the proof to construct (as in [**BS**]) a closed meromorphic 1-form which defines $\mathcal{F}$ in $\overline{\Omega}$. A combination of the existence of such a 1-form with a homotopic argument (reminiscent of an idea of A. Douady) will complete the proof.

We notice that a similar result is true if we assume only that $\mathcal{L}$ has a closed leaf with holonomy h satisfying: $h'(0)$ is a root of 1, h is not periodic (i.e., not linearizable). In that case the model $\mathcal{G}$ is given by the foliation generated by

$$z\frac{\partial}{\partial z} + (z^n + nw)\frac{\partial}{\partial w}$$

and restricted to $\{(z,w) \in \mathbb{C}^2 | |z|^2 + n|w|^2 \leq 1\}$. The proof is virtually the same.

1. Transversely holomorphic foliations with a parabolic closed leaf

Let $\mathcal{L}$ be an oriented transversely holomorphic foliation on a closed 3-manifold M. Let $\gamma \simeq \mathbf{S}^1$ be a parabolic closed leaf of $\mathcal{L}$. The topological dynamics of $\mathcal{L}$ near γ is well-known [**C**]: the holonomy $h: (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of γ is topologically conjugate to $h_0(z) = z + z^2$ (fig. 1). It follows that we can find a (small) tubular neighborhood V of γ with the following properties (fig. 2):

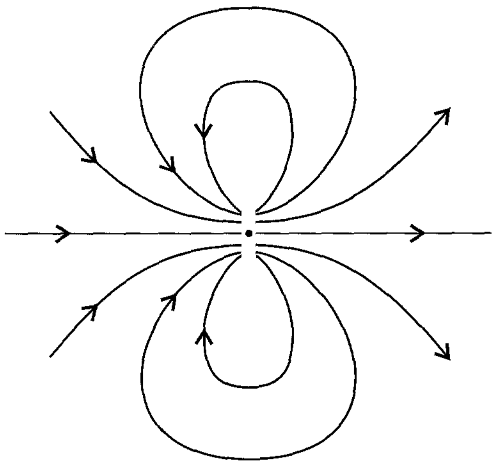

Figure 1

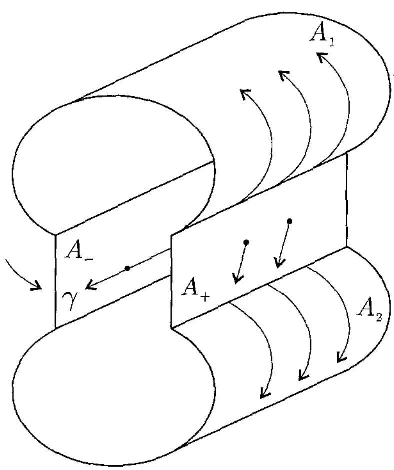

Figure 2

i) ∂V is composed by four smooth closed annuli A_1, A_2, A_+, A_-, intersecting along their boundaries, such that V^c is a compact manifold with boundary and corners;

ii) $\mathcal{L}$ is transverse to A_+ and A_-, it enters into V through A_- and exits from V through A_+;

iii) $\mathcal{L}$ is tangent to A_1 and A_2, and $\mathcal{L}|_{A_j}$ is a foliation by closed intervals, $j = 1, 2$.

A neighbourhood V with these properties will be called *adapted.*

We may apply to the foliation $\mathcal{L}$ restricted to V^c corollary 3.3 of [**BG**], which is an easy corollary to a theorem of Ghys and Gomez-Mont [**GG**]. The conclusion is that every leaf of the foliation $\mathcal{L}|_{V^c}$ starts from A_+ and goes up to A_-, that is, the foliation $\mathcal{L}|_{V^c}$ is conjugate to the foliation on $S^1 \times [0,1] \times [0,1]$ whose leaves are the segments $(\theta, t) \times [0,1]$, $(\theta, t) \in \mathbf{S}^1 \times [0,1]$. Hence every leaf of $\mathcal{L}$ has γ as ω-limit and α-limit set. Moreover, M is a union of two solid tori and hence it is diffeomorphic to $\mathbf{S}^2 \times \mathbf{S}^1$ or $\mathbf{S}^3$ or a lens space.

We now recall some results of Écalle and Voronin (as explained in [**MR**]). Let $h\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a germ satisfying $h'(0) = 1$, $h''(0) \neq 0$. It is formally conjugate, for a suitable and unique $\lambda \in \mathbb{C}$, to $g_\lambda = \exp(X_\lambda)$, where X_λ is the germ of vector field

$$2\pi i \frac{z^2}{1 + \lambda z} \frac{\partial}{\partial z}$$

and $\exp(\cdot)$ denotes the time-one-map. To distinguish between formally conjugate germs one has to consider the *spaces of orbits.* Take a representative of h on some (small) disc $U \subset \mathbb{C}$, and let

$$\mathcal{O}(h) = \frac{U \backslash \{0\}}{z \sim h(z)}.$$

For a good choice of U this space $\mathcal{O}(h)$ is a non-Hausdorff Riemann surface obtained by gluing together two copies of $\mathbb{C}^*$: a neighborhood of 0 (of ∞) of the first copy is glued to a neighborhood of 0 (of ∞) of the second copy via a biholomorphism $\phi_0(\phi_\infty)$. One can assume $\phi_0'(0) = 1$, and then $\phi_\infty'(\infty) = e^{2\pi i \lambda}$ (if h is formally equivalent to g_λ). There is an obvious equivalence relation between spaces of orbits, and two formally conjugate germs are holomorphically conjugate if and only if they have equivalent spaces of orbits [**MR**].

We shall say that $\mathcal{O}(h)$ is *trivial* if ϕ_0 and ϕ_∞ are restriction of a global biholomorphism $\phi: \mathbb{C}^* \to \mathbb{C}^*$ (i.e., there exists a submersion of $\mathcal{O}(h)$ onto $\mathbb{C}^*$); this implies that $\phi'_\infty(\infty) = 1$, i.e. λ is an integer. The space of orbits of g_λ, $\lambda \in \mathbb{Z}$, is trivial, hence by the above classification criterion every germ h with trivial space of orbits is holomorphically conjugate to some g_λ, $\lambda \in \mathbb{Z}$.

Let us return to our transversely holomorphic foliation $\mathcal{L}$.

Lemma 1. *The holonomy h of γ has trivial space of orbits.*

Proof. The space $\mathcal{O}(h)$ can be constructed in the following way. We take two embedded tori T_+ and T_- in $\overline{V}$ such that (fig. 3):

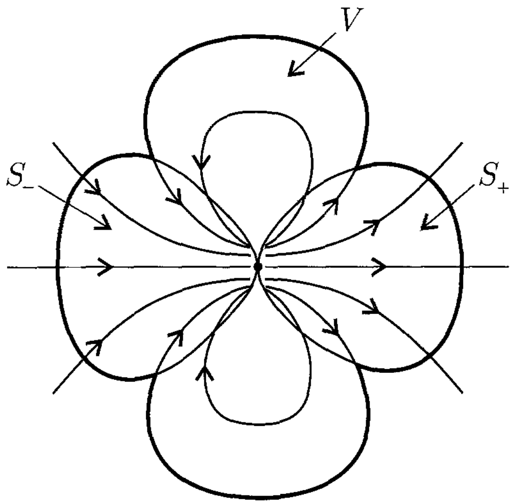

Figure 3

i) $T_+ \cap T_- = \gamma$ and T_+, T_- are tangent along γ; T_+ contains A_+, T_- contains A_-;

ii) T_+ and T_- are boundaries of solid tori S_+ and S_- in $\overline{V}$, with disjoint interiors;

iii) $\mathcal{L}$ is transverse to $T_+\backslash\gamma$ and $T_-\backslash\gamma$, it enters into S_- through $T_-\backslash\gamma$ and exits from S_+ through $T_+\backslash\gamma$.

The foliation $\mathcal{L}$ restricted to $V\backslash\gamma$ gives an identification between $(T_+\backslash\gamma)\backslash A_+$ and $(T_-\backslash\gamma)\backslash A_-$, and the resulting Riemann surface is clearly $\mathcal{O}(h)$ ($T_+\backslash\gamma$ and $T_-\backslash\gamma$ are isomorphic to $\mathbb{C}^*$, and $(T_+\backslash\gamma)\backslash A_+$, $(T_-\backslash\gamma)\backslash A_-$ correspond to neighborhoods of $\{0\}\cup\{\infty\}$ in $\mathbb{C}^*$). But this identification can be holomorphically extended to an identification between $T_+\backslash\gamma$ and $T_-\backslash\gamma$, by the previous remarks about the structure of $\mathcal{L}$ outside V. This shows the triviality of $\mathcal{O}(h)$. □

Hence the holonomy h of γ is holomorphically conjugate to $g_\lambda = \exp(X_\lambda)$, for some unique $\lambda \in \mathbb{Z}$. We now prove that the value of λ completely classifies $\mathcal{L}$. This is based on the non-existence of "holomorphic Dehn twists".

Lemma 2. *Let $\mathcal{L}(\mathcal{L}')$ be an oriented transversely holomorphic foliation on a closed 3-manifold $M(M')$, possessing a parabolic closed leaf $\gamma(\gamma')$ with formal invariant $\lambda(\lambda')$. If $\lambda = \lambda'$ then the two foliations $\mathcal{L}$ and $\mathcal{L}'$ are transversely holomorphically conjugate.*

Proof. The two foliations are conjugate near γ, γ', so that we can find a diffeomorphism $\phi\colon \overline{V} \to \overline{V}'$ between closures of adapted neighborhoods $V \subset M$, $V' \subset M'$, such that $\phi^*(\mathcal{L}'|_{\overline{V}'}) = \mathcal{L}|_{\overline{V}}$ and moreover ϕ is transversely holomorphic. Let $\phi_0\colon \partial V \to \partial V'$ be the restriction of ϕ to the boundary, and consider ∂V and $\partial V'$ as boundaries of V^c and V'^c. Let $A_+ \subset \partial V$, $A'_+ \subset \partial V'$ be the annuli where $\mathcal{L}$, $\mathcal{L}'$ enter into V^c, V'^c. Taking into account the trivial dynamics of $\mathcal{L}$ and $\mathcal{L}'$ in V^c and V'^c, we see that $\phi_0|_{\partial V\backslash A_+}\colon \partial V\backslash A_+ \to \partial V'\backslash A'_+$ obviously extends to $\phi_1\colon V^c \to V'^c$ in such a way that ϕ_1 realizes a transversely holomorphic conjugation between the foliations.

Let us consider the two diffeomorphisms $\phi_0|_{A_+}\colon A_+ \to A'_+$ and $\phi_1|_{A_+}\colon A_+ \to A'_+$. On A_+ and A'_+ there are natural complex structures induced by $\mathcal{L}$ and $\mathcal{L}'$ and preserved by ϕ_0 and ϕ_1. Moreover, by construction, $\phi_0|_{\partial A_+} = \phi_1|_{\partial A_+}$. Hence $\phi_0|_{A_+} = \phi_1|_{A_+}$, because a biholomorphism of an annulus equal to id on the boundary is equal to id everywhere. It follows that ϕ_1 is an extension of $\phi_0|_{\partial V}$, i.e. the initial transversely holomorphic conjugation ϕ extends to all of M. □

It remains to exhibit the models of the classification.

1) We denote by $\mathcal{L}_0$ the foliation on $\mathbf{S}^2 \times \mathbf{S}^1$ obtained by taking the suspension of a parabolic automorphism of the Riemann sphere. It has a parabolic closed leaf with holonomy g_0.

2) We denote by $\mathcal{L}_1$ the foliation on $\mathbf{S}^3$ given by the restriction to

the unit sphere of the holomorphic foliation $\mathcal{G}$ in $\mathbb{C}^2$ generated by the vector field

$$z\frac{\partial}{\partial z} + (z+w)\frac{\partial}{\partial w}.$$

It has a parabolic closed leaf with holonomy g_1 or g_{-1}, depending on the chosen orientation on $\mathcal{L}_1$ (remark that g_λ is conjugate to $g_{-\lambda}^{-1}$).

3) The previous foliation $\mathcal{G}$ in $\mathbb{C}^2$ is invariant under the transformation

$$(z,w) \overset{\sigma_p}{\mapsto} (e^{\frac{2\pi i}{p}}z, e^{\frac{2\pi i}{p}}w), \quad p \in \mathbb{N},$$

which moreover preserves $\mathbf{S}^3$. We denote by $\mathcal{L}_p$, $p \geq 2$, the foliation on $\mathbf{S}^3/\sigma_p = L_p$ (a lens space) obtained by quotienting $\mathcal{L}_1$. This foliation has a parabolic closed leaf with holonomy g_p or g_{-p} (depending on the orientation), because this holonomy is a p-root of g_1 or g_{-1}, i.e. it is $\exp(\frac{1}{p}X_1)$ or $\exp(\frac{1}{p}X_{-1})$ and $\frac{1}{p}X_{\pm 1}$ is conjugate to $X_{\pm p}$.

By the previous results, the *(nonoriented) foliations $\mathcal{L}_p$, $p \geq 0$, exhaust the list of transversely holomorphic orientable foliations on closed 3-manifolds with a parabolic closed leaf.* In particular, on the 3-sphere there is only one such foliation.

This result, and in particular the link between local analysis of the closed leaf and global topology of the ambient manifold, can be better understood in the following way. Let $\gamma_0 \subset \mathbf{S}^2 \times \mathbf{S}^1$ be the closed leaf of $\mathcal{L}_0$. By a Dehn surgery on γ_0 we may construct other foliations with a parabolic closed leaf, but in order to preserve the transverse holomorphicity we need to change the holonomy of γ_0 from g_0 to g_n, where n depends on the index of the Dehn surgery. More precisely, let $(\widehat{M}, \widehat{\mathcal{L}}_0) \xrightarrow{\pi} (\mathbf{S}^2 \times \mathbf{S}^1, \mathcal{L}_0)$ be the blow-up of $\mathcal{L}_0$ along γ_0. On $\partial\widehat{M} \simeq \mathbb{T}^2$ the foliation $\widehat{\mathcal{L}}_0$ is the suspension of a circle diffeomorphism with two fixed points (one attracting and the other repelling). Let $\mathcal{S}$ be a foliation by circles on $\partial\widehat{M}$, transverse to $\widehat{\mathcal{L}}_0$ and such that every circle meets in exactly one point each closed leaf of $\widehat{\mathcal{L}}_0$ on $\partial\widehat{M}$. Every leaf of $\mathcal{S}$ covers γ_0 under π with degree n, for some integer n. If we collapse every leaf of $\mathcal{S}$ to a point we obtain a foliation $\mathcal{L}$ on some manifold M with a closed leaf γ. This $\mathcal{L}$ is transversely holomorphic outside γ, and an easy argument allows to extend this transverse holomorphic structure to all of M. It is then simple to see that the holonomy of γ is exactly g_n.

2. Proof of the theorem

Let $X, \Omega, M, \mathcal{F}, \mathcal{L}$ be as in the statement of the theorem. By the previous results, the holonomy of the parabolic closed leaf γ of $\mathcal{L}$ is holomorphically conjugate to the holonomy of $\widehat{\gamma} = \{(z,w) \in \mathbb{C}^2 | z = 0, |w| = 1\}$ with respect to the foliation $\widehat{\mathcal{L}}$ induced by $\mathcal{G}$ on the unit sphere. Let $\Gamma \in \mathcal{F}$ be the leaf containing γ, $\widehat{\Gamma} \in \mathcal{G}$ the leaf containing $\widehat{\gamma}$ (i.e., $\widehat{\Gamma} = \{z = 0, w \neq 0\}$).

A sufficiently small neighborhood of γ in X is biholomorphic to a neighborhood of $\hat{\gamma}$ in $\mathbb{C}^2$, the biholomorphism sending a piece of Γ to a piece of $\hat{\Gamma}$ (see, e.g., [**A,** §**27**]. It follows that near γ, as near $\hat{\gamma}$, there exists a holomorphic foliation by discs transverse to $\mathcal{F}$.

Let f_0 be a biholomorphism between a neighborhood of γ in Γ and a neighborhood of $\hat{\gamma}$ in $\hat{\Gamma}$. Because γ and $\hat{\gamma}$ have the same holonomy, using the above foliations transverse to $\mathcal{F}$ and $\mathcal{G}$ near γ and $\hat{\gamma}$ we can suspend f_0 to a biholomorphism f between a neighborhood of γ in X and a neighborhood of $\hat{\gamma}$ in $\mathbb{C}^2$, sending $\mathcal{F}$ to $\mathcal{G}$.

Lemma 3. *There exists a closed meromorphic* 1*-form* ω*, defined on a neighborhood of* $\overline{\Omega}$*, which defines* $\mathcal{F}$ *and whose polar divisor* $(\omega)_\infty$ *intersects* M *in* γ*.*

Proof. The foliation $\mathcal{G}$ can be given by the closed meromorphic 1-form $\frac{1}{z^2}[(z+w)dz - zdw]$, whose polar divisor is $\{z = 0\}$. This 1-form can be pulled back by the previous biholomorphism f, giving a 1-form ω near γ, with polar divisor contained in Γ.

The dynamics of $\mathcal{F}$ near M (i.e. the dynamics of $\mathcal{L}$) allows to extend ω to a neighborhood of M: ω is outside Γ the differential of a "multivalued" holomorphic function, constant on the leaves, it is then sufficient to extend such a function constantly on leaves and then to take the differential of the extension. This is coherent because the nontrivial dynamics of $\mathcal{L}$ is concentrated on γ.

Finally, ω is extended from $\partial\Omega$ to Ω via a Levi-type theorem: we may write

$$\omega = \sum_{j=1}^{N} f_j \omega_j$$

where ω_j are holomorphic 1-forms on X and f_j are meromorphic functions near $\partial\Omega$, and then it is sufficient to extend every f_j to all of Ω, which is possible thanks (e.g.) to [**I, prop. 1.1**] and the classical Levi's theorem [**S**] (remark that the envelope of holomorphy of a neighborhood of $\partial\Omega$ contains Ω, by Hartog's kugelsatz).

By construction, the polar divisor $(\omega)_\infty$ intersects M exactly along γ. $\square$

We set $L = (\omega)_\infty \cap \overline{\Omega}$. It is an analytic curve, invariant by $\mathcal{F}$.

Lemma 4. *There exists a holomorphic vector field v on a neighborhood of $\overline{\Omega}$ which generates $\mathcal{F}$.*

Proof. Because X is Stein, the tangent bundle $T\mathcal{F}$ to $\mathcal{F}$ (see [**G**] for definition and properties) is holomorphically classified by $H^2(X, \mathbb{Z})$. Because this cohomology group is trivial, this line bundle is holomorphically trivial. A nonvanishing section of $T\mathcal{F}$ gives a vector field on X generating $\mathcal{F}$. $\square$

Let $E \subset TX|_M$ denote the bundle of complex lines tangent to M: for every $p \in M$, E_p is the unique complex line in $T_pM \subset T_pX$. Let $\mathbf{S}$ denote the space of continuous sections $s: M \to TX|_M$ such that $s(p) \notin E_p \quad \forall p \in M$. Because E_p has (real) codimension 2 in T_pX, and because $M \simeq \mathbf{S}^3$, this space is homotopically equivalent to $C^0(\mathbf{S}^3, \mathbf{S}^1)$, in particular every two sections in $\mathbf{S}$ are homotopic in $\mathbf{S}$.

It follows that if $w_1, w_2: \overline{\Omega} \to TX|_{\overline{\Omega}}$ are sections such that $w_j|_M \in \mathbf{S}$ then w_1 and w_2 have the same Poincaré-Hopf index in $\overline{\Omega}$. Clearly this common index must be 1 (take a radial field on $\overline{\Omega}$). Hence if $w: \overline{\Omega} \to TX|_{\overline{\Omega}}$ is holomorphic and $w|_M \in \mathbf{S}$ (equivalently, the holomorphic foliation generated by w is transverse to M) then w has in $\overline{\Omega}$ exactly one singularity, of multiplicity one (its eigenvalues are both different from zero).

Observe that for every $p \in \gamma$ we have that $T_p\Gamma$ is transverse to E_p. If $s \in \mathbf{S}$, then we may project $s|_\gamma$ to $T\Gamma|_\gamma$ along $E|_\gamma$, obtaining a nonvanishing section of $T\Gamma|_\gamma$. The homotopy class of this nonvanishing section is again independent of s, so that this nonvanishing section is homotopic to one transverse to γ.

Lemma 5. *The polar divisor L is a smooth disc, which contains a singularity q of $\mathcal{F}$ of the type $z\frac{\partial}{\partial z} + (w+z)\frac{\partial}{\partial w}$.*

Proof. Recall that there is a holomorphic vector field v near $\overline{\Omega}$ which generates $\mathcal{F}$; v has in Ω only one singularity q, of multiplicity 1, and it is tangent to L. Singularities of L are also singularities of v, hence either $q \notin L$ (and L is smooth, $v|_L$ is without singularities) or $q \in L$ and, near q, L coincides with a union of k separatrices of q.

The former case cannot occur: L would be a compact orientable surface with $\partial L \simeq \mathbf{S}^1$ and v would define a nonsingular vector field on L transverse to ∂L (up to homotopy, see remarks above). Contradiction with Poincaré-Hopf formula.

Hence $q \in L$. Let $\widetilde{L}$ be the normalization of L, it is smooth and it supports a vector field $\widetilde{v}$ transverse to $\partial L \simeq \mathbf{S}^1$ (up to homotopy) and with k singularities in correspondence of the points which project to q under $\widetilde{L} \to L$. Each singularity has index 1, because q has multiplicity 1, and Poincaré-Hopf formula again ($\chi(\widetilde{L}) = k$) implies: $\widetilde{L}$ is a disc and $k = 1$.

Hence L near q coincides with a single separatrix, whose holonomy is equal to the holonomy of γ because $L\backslash\{q\} \simeq \overline{\mathbb{D}}^2\backslash\{0\}$. There are only two possibilities [**A**], [**MR**]:

i) q is in the Siegel domain and hence of the type

$$z(1+nzw)\frac{\partial}{\partial z} - w(1+(n-1)zw)\frac{\partial}{\partial w},$$

and L coincides locally with $\{z=0\}$;

ii) q is in the Poincaré domain and hence of the type

$$z\frac{\partial}{\partial w} + (w+z)\frac{\partial}{\partial w},$$

and L coincides locally with $\{z=0\}$.

But the case i) cannot occur, because we know that $\mathcal{F}$ outside L is given by a holomorphic closed 1-form, which does not happen in case i) (there would be holonomy outside L). This completes the proof. □

By lemma 5 we may choose a small ball B around q such that ∂B is transverse to $\mathcal{F}$ and $\mathcal{F}|_{\overline{B}}$ is conjugate (via a transversely holomorphic diffeomorphism) to $\mathcal{G}|_{\{|z|^2+|w|^2\leq 1\}}$. In particular $\mathcal{F}|_M$ and $\mathcal{F}|_{\partial B}$ are conjugate, and we now prove that $\mathcal{F}|_{\overline{\Omega}\setminus B}$ is a trivial cobordism between $\mathcal{F}|_M$ and $\mathcal{F}|_{\partial B}$, which will complete the proof of our theorem.

Let $F \in \mathcal{F}|_{\overline{\Omega}\setminus B}$ be a leaf different from the closed annulus $L\cap(\overline{\Omega}\setminus B)$. This leaf accumulates only on $L\cap(\overline{\Omega}\setminus B)$, by the existence of the closed meromorphic 1-form ω. Remark that this 1-form ω, holomorphic outside L, can be integrated to give a holomorphic first integral $H: \bar{\Omega}\setminus L \to \mathbb{C}$: set

$$H(p) = \exp(\frac{2\pi i}{T}\int_{p_0}^{p}\omega),$$

where T is the period of ω around L, p_0 belongs to $\bar{\Omega}\setminus L$, and the integral is done on any path in $\bar{\Omega}\setminus L$ joining p_0 to p. We know the precise structure of $\mathcal{F}$ near L. It follows that F is a noncompact surface whose boundary is a finite union of lines, contained in $M\cup\partial B$. The generator v of $\mathcal{F}$ gives a nonsingular vector field on F, whose behaviour (up to homotopy) is known near each connected component of ∂F (by a simple variation of the homotopical considerations previously used when analyzing L) and outside a compact set (because F accumulates on L), see fig. 4.

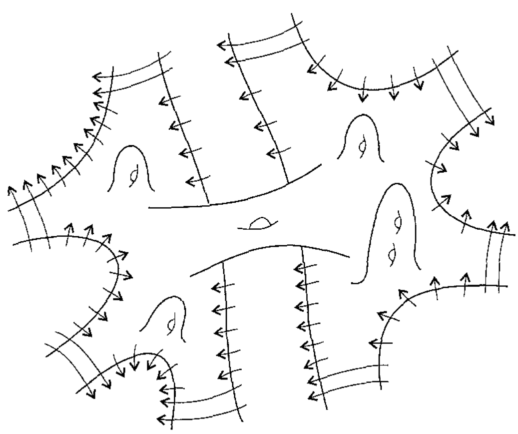

Figure 4

It follows again by Poincaré-Hopf formula that F is diffeomorphic to $\mathbb{R} \times [0,1]$, with $\mathbb{R} \times \{0\} \subset \partial B$ and $\mathbb{R} \times \{1\} \subset M$. It is then clear that $\mathcal{F}|_{\overline{\Omega} \setminus B}$ is a trivial cobordism.

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